Limit Evaluation Techniques – Limits of Infinity

The example of $\frac{1}{x}$

Recall that we briefly mentioned this in the page on substitution. The most common way of getting a limit **of** infinity is to have something like: $\lim_{x \to 0+} \frac{1}{x}$.

When substitution works, there's no thinking involved. But in this example we have a problem. The function is not even defined at x=0. True limit thinking is needed.

True limit thinking

What do I mean by that? We need to understand two things:

* When we are taking a limit we are not (yet) interested in what the value of the function is at the actual x-value. We want to know what the value of the function "**should be**" (so we can possibly re-define it to make it continuous).

* We know we can get arbitrarily close to a given x-value. We want to **know** if the corresponding function value is ALSO getting arbitrarily close to a specific real number. Notice the emphasis on 'know'. Many times we can conjecture what the limit is based on numerical examination. That is only because we choose easy problems that have integer answers. We learned centuries ago that the world is much more complicated than one that an integer-only model matches.

The problem with 0 in the denominator

Zero sits on the Great Divide between all the negative numbers and all the positive numbers. 'Getting closer and closer to 0' also means getting closer and closer to small numbers that have an opposite sign. And even this behavior is usually OK as long as I am in the numerator. The problem is in the denominator – I could be throwing the whole fraction back and forth between very large positive numbers and very large negative numbers.

One side at a time

The way I handle this is to define left hand limits (limits from the left or negative side) and right hand limits (limits from the right or positive side). I do this:

* because that's the way we defined 'limit' (meaning two-sided limit), and
* because I also restrict the x-values to positive-only or negative-only real numbers.

So – **any** time you are dealing with a limit that appears to have a zero-ish thing in the denominator, consider the limit from each side.

If they are not equal, the limit 'does not exist' (DNE).

If they are both equal to $+\infty$, then write that as the limit. (It provides more information than DNE, but remember that since $+\infty$ is not a number, 'does not exist' is also meant. In other words, for a situation like this – on a multiple choice question that has responses of $+\infty$ AND 'does not exist', **then the best answer** is $+\infty$!)

If they are both equal to $-\infty$, then write that as the limit. (Same reasoning as above).

Practice

There is real thinking involved in these problems, even though there are only 3 possible answers (does not exist, +∞, -∞). Or something else (just to be twisted).

1. $\lim_{x\to 0+}\frac{1}{x}$; $\lim_{x\to 0-}\frac{1}{x}$; and of course $\lim_{x\to 0}\frac{1}{x}$

2. $\lim_{x\to 0+}\frac{1}{x^2}$; $\lim_{x\to 0-}\frac{1}{x^2}$; and of course $\lim_{x\to 0}\frac{1}{x^2}$

3. $\lim_{x\to 0+}\frac{1}{x^3}$; $\lim_{x\to 0-}\frac{1}{x^3}$; and of course $\lim_{x\to 0}\frac{1}{x^3}$. You get the idea. But there are also:

4. $\lim_{x\to 0+}\frac{1}{\sin x}$; $\lim_{x\to 0-}\frac{1}{\sin x}$; and of course $\lim_{x\to 0}\frac{1}{\sin x}$. Watch out for things like:

5. $\lim_{x\to \frac{\pi}{2}+}\tan x$; $\lim_{x\to \frac{\pi}{2}-}\tan x$; and of course $\lim_{x\to \frac{\pi}{2}}\tan x$.

6. $\lim_{x\to 3+}\frac{1}{x-3}$; $\lim_{x\to 3-}\frac{1}{x-3}$; and of course $\lim_{x\to 3}\frac{1}{x-3}$.

7. $\lim_{x\to 1+}\frac{x}{\ln x}$; $\lim_{x\to 1-}\frac{x}{\ln x}$; and of course $\lim_{x\to 1}\frac{x}{\ln x}$.

8. $\lim_{x\to 0+}x^{-2/3}$; $\lim_{x\to 0-}x^{-2/3}$; and of course $\lim_{x\to 0}x^{-2/3}$.